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DIFFRACTION OF SPHERICAL WAVES BY A CYLINDRICAL CAVITY REINFORCED WITH A STIFF SHELL†

YU. B. NAZARENKO

Moscow Region

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A new approach based on variational principles is proposed. It enables one to determine the dynamic displacements of a spherical shell with an arbitrary contour on the basis of the solution of the problem for a circular shape [1]. The use of this approach, in conjunction with the small-parameter method [2], considerably increases the accuracy when estimating the stress distribution. When a point source, which satisfies the Sommerfeld radiation condition, is far away, the solution is extended to the case of the diffraction of spherical waves by a cylindrical cavity reinforced with a soldered on, stiff shell.

1. THE DIFFRACTION OF SPHERICAL WAVES BY THE CYLINDRICAL CAVITY OF CIRCULAR PROFILE

WE WILL seek a solution of the equation of motion for the case of an isotropic elastic body using the scalar potential φ and the vector potential ψ which are connected to the displacements by the Helmholtz relationship [3].

Let us introduce a cylindrical system of coordinates, with the z axis directed along the axis of the cylindrical cavity, and let us represent the equation of motion in terms of the potentials in the form

$$\nabla^2\varphi + \frac{1}{C_1^2}\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial^2\varphi}{\partial z^2} = 0, \quad \nabla^2\psi_l + \frac{1}{C_2^2}\frac{\partial^2\psi_l}{\partial t^2} + \frac{\partial^2\psi_l}{\partial z^2} = 0 \tag{1.1}$$

where φ is the scalar potential, ψ_i are the projections of the vector potential on to the coordinate axes, and C_1 and C_2 are the longitudinal and transverse velocities of wave propagation.

We will represent the incident wave in a plane normal to the axis of the cavity. The wave has a rectilinear front and varies along the z axis in a known way. Neglecting the component of the mechanical action along the axis of the cavity, we take the scalar potential of the incident wave, which satisfies these conditions and Eq. (1.1), in the form

$$\varphi = \sum_{n=0}^{\infty} \varphi_0 \gamma i^n I_n(\alpha r)\cos\epsilon \exp[-i(Kz+\omega t)]$$
$$\alpha = \sqrt{\upsilon^2 + K^2}, \quad \epsilon = n(\theta - \chi). \tag{1.2}$$

The longitudinal potential φ^* and the transverse potential ψ^* of the reflected waves are specified by the series

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$$\begin{Bmatrix} \varphi^* \\ \psi^* \end{Bmatrix} = \sum_{n=0}^{\infty} \begin{Bmatrix} A_n \\ B_n \end{Bmatrix} H_n\left(\begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} r\right) \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} \epsilon \exp\left[-i(Kz+\omega t)\right],$$
$$\beta = \sqrt{w^2 + K^2} \tag{1.3}$$

where φ_0 is the maximum amplitude of the dynamic action, I_n and H_n are Bessel and Hankel functions, A_n and B_n are constant coefficients, determined from the boundary conditions, v and w are wave numbers, defined by the relationships $v=\omega/C_1$ and $w=\omega/C_2$, ω is the angular frequency, K is a separation constant, χ is the angle between the direction of propagation of the longitudinal wave and the vertical axis coinciding with the origin from which the angle θ is measured, and $\gamma=1$ when $n>0$ and $\gamma=0.5$ when $n=0$.

In order to estimate the stress–strain state of the medium in the spatial formulation, we will adopt the condition that the radiation source is sufficiently far removed from the cavity and consider the scalar potential φ and the projection of the potential ψ on to the z-axis. As in the planar formulation of the problem, we put the projections of the vector potential on to the other axes equal to zero in view of their negligible effect. This follows from the general solution of the system of differential equations in a spatial elastic body by the small-perturbation method, according to which this effect is determined by the solution of the zeroth approximation, which does not take into account terms with small values of the coefficients and the perturbations in the solution, which is determined using a known method when the corresponding terms are taken into account.

Since, in the case of the above-mentioned projections of the vector potential, the zeroth solution corresponds to a plane wave and is equal to zero, the perturbations for them are also equal to zero. We shall therefore only consider perturbations for those potentials which also had solutions in the case of the planar formulation of the problem.

The stresses in the body when a wave is reflected by the cylindrical cavity are determined by the sum of the potentials of the incident wave (1.2) and the reflected wave (1.3). Calculating them using a well-known technique, we determine the components of the stressed state in the $r\theta$ plane

$$\sigma_r = \sum_{n=0}^{\infty} \exp\left[-i(Kz+\omega t)\right] \cos\epsilon\{\lambda(1-K^2)\alpha^2[-2\gamma\varphi_0 i^n f_1 - A_n f_2] + $$
$$+\mu[4\gamma\varphi_0 i^n F_1 + 2A_n F_2 - 2B_n F_3]\}$$
$$\sigma_\theta = \sum_{n=0}^{\infty} \exp\left[-i(Kz+\omega t)\right] \cos\epsilon\{\lambda(1-K^2)\alpha^2[-2\gamma\varphi_0 i^n f_4 - A_n f_6] - $$
$$-\mu[4\gamma\varphi_0 i^n F_4 + 2A_n F_5 - 2B_n F_3]\} \tag{1.4}$$
$$\sigma_{r\theta} = \sum_{n=0}^{\infty} \exp\{-i(Kz+\omega t)\}\ \mu\sin\epsilon[4\varphi_0 i^n L_4 + 2A_n L_5 + B_n F_6]$$

$$F_{1,2} = [\frac{(n+n^2)}{r^2} - \alpha^2]f_{1,2} - \frac{\alpha}{r} g_{1,2},\quad F_3 = \frac{(n+n^2)}{r^2} f_6 - \frac{n\beta}{r} g_6$$

$$F_4 = F_1 + \alpha^2 f_5,\quad L_{4,5} = F_{4,5} + \frac{\alpha}{r}(1-n) g_{4,5}$$

$$F_6 = \frac{2\beta}{r} g_6 + [\beta^2 - \frac{2(n+n^2)}{r^2} f_6,\quad f_1 = f_4 = I_n(\alpha r),\quad f_2 = f_5 = H_n(\alpha r),$$

$$g_1 = g_4 = I_{n-1}(\alpha r)$$
$$g_2 = g_5 = H_{n-1}(\alpha r),\quad f_6 = H_n(\beta r),\quad g_6 = H_{n-1}(\beta r)$$

Here λ and μ are Lamé constants. The coefficients A_n and B_n in expressions (1.4) are determined from the boundary conditions on the circular contour of the cavity, while the separation constant K is determined from an approximation of the spherical wave front which satisfies the Sommerfeld radiation condition [4] along the cavity

$$\varphi = \frac{\varphi_0^*}{S} \exp\{ i\omega(S/C_1 - t)\} \tag{1.5}$$

where S is the distance from the point source to the cavity and φ^* is the maximum pressure in the wave front.

2. THE DIFFRACTION OF SPHERICAL WAVES BY A CYLINDRICAL CAVITY WITH A NON-CIRCULAR CONTOUR

Let us represent the contour of the cavity in the form

$$r = r_0 + \sum_{k=-\infty}^{k=\infty} S_k \exp(ik\theta) \tag{2.1}$$

Here r_0 is the reduced radius of the contour of the cut-out and S_k is the amplitude of the deviation of the contour of the k-harmonic in the Fourier expansion from a circular shape. When the deviations in the contour from a circular shape are insignificant, the solution of the problem may be reduced to a polar-symmetric problem using the small-parameter method [2]. The problem is solved by linearizing the stress distribution within the limits of the deviation of the contour from a circular shape, transfer of the boundary conditions on to the circular contour, and is represented in the form of the stresses of the zeroth approximation $\sigma^{(0)}$ and, also, the perturbations in the stresses $\sigma^{(1)}$ due to the deviations in the contour from a circular shape. These stresses have been presented for the condition of plane deformation [5], but can also be used to describe the stressed state of any section normal to the axis of a cavity in the diffraction of spherical waves. In this case, the boundary conditions take the form

$$\begin{aligned}
\sigma_r^{(1)} = \sum_{n=0}^{\infty} \cos\epsilon \Big\{ & \lambda(1-K^2)\alpha^2[2\gamma\varphi_0 i^n (g_1\alpha - \frac{n}{r} f_1) + \\
& + A_n(g_2\alpha - \frac{n}{r} f_2)] - 2\mu[2\gamma\varphi_0 i^n(\xi f_1 - \zeta f_1 + \eta g_1) + A_n(\xi f_2 - \zeta f_2 + \eta g_2) - \\
& - B_n F_7] \Big\} \sum_{k=-\infty}^{k=\infty} S_k \exp[i(k\theta - \omega t)] \\
\sigma_{r\theta}^{(1)} = \Big\{ \sum_{n=0}^{\infty} \frac{ik}{r} & \cos\epsilon\, [\lambda(1-K^2)\alpha^2(-2\gamma\varphi_0 i^n f_1 - A_n f_2) + \\
& + 2\mu(-2\gamma\varphi_0 i^n F_4 - A_n F_5 + B_n F_3)] - \sum_{n=0}^{\infty} \mu \sin\epsilon [4\varphi_0 i^n L_1 + \\
& + 2A_n L_2 + B_n F_8] \Big\} \sum_{k=-\infty}^{k=\infty} S_k \exp[i(k\theta - \omega t)]
\end{aligned} \tag{2.2}$$

$$F_7 = [\frac{n\beta^2}{r} - \zeta] f_6 + \frac{3ng_6\beta}{r^2}, \quad F_8 = [2\zeta - \frac{\beta^2(n+2)}{r}] f_6 +$$

$$+ \beta[\beta^2 - \frac{2(n^2+2)}{r^2}] g_6, \quad \xi = \frac{\alpha^2(1+n)}{r}, \quad \zeta = \frac{n^3 + 3n^2 + 2n}{r^3},$$

$$\eta = \alpha[\frac{n^2+2}{r^2} - \alpha^2], \quad L_{1,2} = [\frac{n\alpha}{r} - \zeta] f_{1,2} + \frac{3n\alpha g_{1,2}}{r^2}$$

3. REINFORCEMENT OF THE CUT-OUT WITH A STIFF RING

When there is close contact between the stiff ring and the plate, we obtain the solution by reducing the contact problem of the interaction of two media with different mechanical characteristics to a homogeneous problem by an equivalent change in the ring parameters. however, the use of the small-parameter method in order to change to polar-symmetric boundaries is confined to the case when there are insignificant deviations in the contour of the opening from a circular shape.

In order to remove this constraint, we propose to use the principle of an equivalent change in the rigidity and load on changing from a non-circular ring to a circular ring [1] using the Ritz–Timoshenko approach, and also to make use of the condition that the minima of the potential energies of the systems are fairly close in the case of non-circular and circular rings when the contour only slightly deviates from a circular shape.

Taking account of the fact that, by the variational principle, the true displacements must correspond to minimum potential energy, we determine them for a non-circular ring from the solution for a circular ring subject to the condition that their energies are equal.

Let us consider the potential energy of a non-circular ring with coordinates defined by expression (2.1) in an arbitrary interval $\theta_1 < \theta < \theta_2$, in the form

$$\int_{\theta_1}^{\theta_2} \left[\frac{E_r I}{2R_0^2\rho^2}\left(\frac{d^2U}{d\theta^2} + U\right)^2 - p\frac{dU}{d\theta} - qU\right]\left(R_0 + \sum_{k=-\infty}^{k=\infty} S_k \exp(ik\theta)\right) d\theta \tag{3.1}$$

and, also, that for a circular ring with a radius of the median line, R_0

$$\int_{\theta_1}^{\theta_2} \left[\frac{E_r I_c}{2R_0^4}\left(\frac{d^2U}{d\theta^2} + U\right)^2 - p_c\frac{dU}{d\theta} - q_c U\right] R_0 d\theta \tag{3.2}$$

The subscript r refers to the case of a ring, and the subscript c refers to the case of a circular ring.

When $kS_k \ll R_0$ and small quantities are neglected, we establish the condition for the equivalent transition to a circular ring due to a change in the external load

$$\begin{Bmatrix} p_c \\ q_c \end{Bmatrix} = \begin{Bmatrix} p \\ q \end{Bmatrix}\left(1 + \frac{1}{R_0}\sum_{k=-\infty}^{k=\infty} S_k \exp(ik\theta)\right) \tag{3.3}$$

and the stiffness of the ring at each of its points

$$E_r I_c = E_r I(1 + T)$$
$$T = \sum_{k=-\infty}^{k=\infty}\left[(2k^2 - 1)\frac{S_k}{R_0}\exp(ik\theta) - \frac{k^2 S_k^2}{R_0^2}\exp(2ik\theta)\right] \tag{3.4}$$

Here p and q are the radial and tangential components of the external load acting on the ring, $E_\gamma I$ is the stiffness of the non-circular ring, ρ is its radius of curvature, and U are the tangential displacements.

Let us consider the solution of the contact problem in the case of a uniform component of the loading. We obtain the zeroth-approximation stresses in the medium in the case of circular ring contours and non-circular cut-out contours taking into account St Venant's principle of continuity. For this purpose, we will consider a ring with a modulus of elasticity which is the same as that of the medium due to a change in its thickness, when constancy of the stiffness under uniaxial loading is maintained, and let us also compensate for the gap in the cavity contour due to the change in the internal boundary of the ring. The coordinates of the internal contour of the ring are then determined from the expression

$$r = m + \sum_{k=-\infty}^{k=\infty} \frac{S_k \exp(ik\theta)}{m}, \quad m = r_2 - \frac{(r_2 - r_1)E_r}{E} \tag{3.5}$$

Here r_1 and r_2 are the internal and external reduced radii of the ring, E_γ is its modulus of elasticity, and E is the modulus of elasticity of the body.

Under uniform loading, we determine the stresses in the zeroth approximation and for the perturbations using the expressions in [5], on imposing boundary conditions (2.2) taking account of (3.5).

We obtain the solution of the contact problem in the case of a non-uniform load by changing to a homogeneous medium by means of an equivalent change in the ring parameters, when its rigidity remains unchanged under different loading conditions, and by compensating for the gap in the cavity contour. In this case, we determine the stresses in the zeroth approximation using expressions (1.4) when the value of the reduced radius of the internal circular contour is

$$r = r_2 - (r_2 - r_1)(E_r/E)^{1/3} \tag{3.6}$$

We then find the expressions in the stresses subject to boundary conditions (2.2) using the expressions in [1] for the deviations of the contour governed by the dependence

$$s = -W\left[\frac{T(r_2 - r_1)}{3} - \left(\frac{E}{E_r}\right)^{1/2}\left(\frac{r_1}{r_2}\right)^{2/3} \sum_{k=-\infty}^{k=\infty} S_k \exp(ik\theta)\right] \tag{3.7}$$

Here, $W = (E_\lambda / E)^{1/3}$ in the case of boundary conditions (2.2) which are non-uniform in a circumferential direction and $W = E_r / E$ in the case of uniform boundary conditions.

We obtain the stress distribution in a real ring by solving the force equilibrium differential equation for a circular ring in the quasistatic formulation when the stress distribution on its external surface is established taking account of the changes in the stresses using expression (3.3) and, in the case of variable stiffness of the ring (3.4), using the expressions in [1].

4. RESULTS OF THE INVESTIGATIONS

Let us estimate the effect of the closeness of a finite source which radiates spherical waves with frequency $\omega = 400$ rad/s into a cylindrical cavity with radius $r_0 = 1.0$ m, located in a medium with modulus of elasticity $E = 2.4 \times 10^3$ MPa and Poisson's ratio equal to 0.2.

In order to establish the parameters for an approximation of the incident wave (1.2) which satisfies the Sommerfeld radiation condition (1.5) we expand expressions (1.2) and (1.5) in Taylor's series with respect to the variable z and consider only those terms which turn out to have a substantial effect. Then, on maintaining the condition $K^2 \ll \omega^2 / C_2^2$ and, also the constraints $LS \ll 1$ and $LK \ll 2$, we obtain

$$K = \left[6 - 12 \cdot \left(\frac{1}{4} - \frac{L^4 \omega^2}{144 S^2 C_1^2} + \frac{L^2}{6S^2}\right)^{1/2}\right]^{1/2} / L \tag{4.1}$$

Here, S is the smallest distance from the point source to the cylindrical cavity.

We determine the length of the cavity L, over which the boundary conditions are satisfied, taking account of the fact that there is no boundary effect in the initial section $z = 0$. Then, on solely taking account of the cosine expansion along the z-axis in expression (1.2) and considering a section of the cavity in the range $-\pi/2K < \theta < \pi/2K$, we establish that the length is equal to 13 m (Fig. 1). The dependence of the stress concentration f on the length of the cavity L over which the approximation of the external action is made is represented by the solid line.

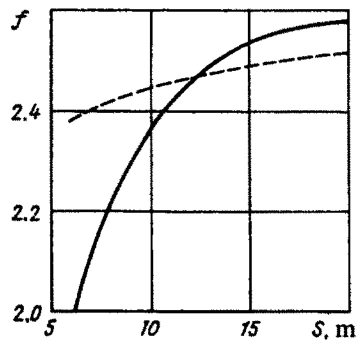

FIG. 1.

The estimate of the effect of the closeness of the source of radiation of waves into a cylindrical cavity is shown for the initial data which have been adopted and the dependence of f in the most loaded sections, $\theta = \pm\pi/2$, on the remoteness of the source S (the dashed line) has been established.

The solution of the problem of the diffraction of spherical waves by a cylindrical cavity, which has been presented above, enables one to evaluate the stress–strain state over a wide range of distances of the point source and to establish the limits of applicability of the plane solution.

The proposed method of reducing contact problems of the concentration of stresses in cylindrical cavities with non-circular contours to polar-symmetric boundary conditions by an equivalent change in the stiffness of the shell and its loading parameters is not subject to rigorous constraints regarding the magnitudes of the deviations of the shell contour from a circular shape and, moreover, on their derivatives, as in Pal'mov's small-parameter method. This approach is quite compatible with the small-parameter method [2] and, which is the prime consideration, the error in estimating the state of stress of a body using the small-parameter method will be reduced by compensating for the deviation of the contour due to the non-uniform stiffness of the equivalent circular ring when the two methods are used. Hence, we may expect fairly good accuracy and reliability for the proposed approach to solving problems of the concentration of stresses in cylindrical cavities with soldered on shells.

REFERENCES

1. NAZARENKO Yu. B., Diffraction of decaying waves in mines in the case of plastic performance of the marginal zone of a body. *Izv. Vuz. Gorn. Zhurn.* **7**, 31–36, 1988.
2. PAL'MOV V. A., Stressed state close to the rough surface of elastic bodies. *Prikl. Mat. Mekh.* **27**, 5, 963–969, 1968.
3. PAO YIH-HSING, Dynamical stress concentration in an elastic plate. *Trans. ASME Ser. E, J. Appl. Mech.* **29**, 2, 299–305, 1962.
4. KUPRADZE V. D., *Fundamental Problems of the Mathematical Theory of diffraction.* Gl. Red. Obshche-Tekhn. Lit., Leningrad and Moscow, 1935.
5. NAZARENKO Yu. B. Analysis of mining props under the action of a dynamic load. *Izv Vuz. Gorn. Zhurn.* 9, 22–27, 1986.

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